Vectorspace

Abstract Systems. Binary Operations and Relations. Introduction to Groups and Fields. Vector Spaces and Subspaces. Linear Independence and Dependence of Vectors. Basis and Dimensions of a Vector Space. Change of basis. Homomorphism and Isomorphism of Vector Spaces. Linear Transformations. Algebra of Linear Transformations. Non- singular Transformations. Representation of Linear Transformations by Matrices.

Group :→ A non-empty set S of elements $a, b, c, \ldots$ forms a group with respect to the binary operation $*$, if the following properties hold :

1. For every pair a and $b \in S$, $a * b$ is in S (closure law).
2. For any three elements $a, b, c \in S$,
   $$a * (b * c) = (a * b) * c \quad \text{holds (associative law).}$$
3. There exists in S an element i, called a left identity, such that
   $$i * a = a, \text{ for every } a \in S.$$ the solution x is called left inverse of a.
4. For each a in S, the equation $x * a = i$ has a solution x in S.

Example :→ consider the set of integers (+ve, −ve, zero)

$$Z = \{\ldots\ldots -2, -1, 0, 1, 2, \ldots\ldots\}$$

on which the binary operation is applied. For any $a, b, c \in Z$ we have,

1. $a + b \in Z$ (closure)
2. $(a+b)+c = a+(b+c)$ {associativity}
3. $0 + a = a$ {0 is the left identity element}
4. $(-a) + a = 0$ {left inverse of a is $-a \in Z$}.

Note: → A non-empty set G equipped with one or more binary operations is called an algebraic structure.

→ $(G, *) \Rightarrow$ algebraic structure with one binary operation.

→ $(G, +, \cdot) \Rightarrow$ algebraic structure with two binary operation.

Internal composition :→ Let S be any non-empty set. If $a * b \in S$ $\forall\, a, b \in S$ and $a * b$ is unique, then $*$ is said to be an internal composition in the set S.

External composition :→ Let V and F be any two non-empty sets if $a \circ \alpha \in V$, $\forall\, a \in F$ and $\forall\, \alpha \in V$ and $a \circ \alpha$ be unique then '$\circ$' is said to be an external composition in V over F. $[V(F)]$.

Vector Space :→ Let $(F, +, \cdot)$ be a field. Then a non-empty set V is called vector space over the field F, if in V there be defined an internal composition $*$ and an external composition '$\circ$' over F such that, for all $a, b \in F$ and all $\alpha, \beta \in V$,

1. $(V, *)$ is an abelian group.
2. $a \circ [\alpha * \beta] = [a \circ \alpha] * [a \circ \beta]$
3. $[a + b] \circ \alpha = [a \circ \alpha] * [b \circ \alpha]$
4. $[a \cdot b] \circ \alpha = a \circ [b \circ \alpha]$
5. $1 \circ \alpha = \alpha$, the unit scalar $1 \in F$.

1 is the multiplicative identity of the field F

Vector Sub-space :→ Let V be a vector space over the field F. A non-empty sub-set W of V is called vector subspace or linear sub space or simply sub-space of V, if W itself be a vector over F with respect to the same compositions as defined in V.

The whole vector space 'V' is a sub-space of V and the sub-set consisting of zero vector alone is also a sub-space of V, called the zero sub-space of V. These two are called improper sub-space, while the other sub-spaces are called proper sub-spaces.

Theorem 1 The necessary and sufficient condition for a non-empty sub-set W of a vector space V over F to be a sub-space of V is that W is closed under vector addition and scalar multiplication in V.

$\therefore$ If $\alpha, \beta \in W \Rightarrow \alpha + \beta \in W$,

$a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$.

Example :→ Show that the set W of ordered triad $(a_1, a_2, 0)$, where $a_1, a_2 \in F$, a field, is a sub-space of V_3 over F.

Sol^n :→ Let $\alpha = (a_1, a_2, 0)$ and $\beta = (b_1, b_2, 0)$ belong to W, where $a_1, a_2, b_1, b_2 \in F$. If a, b be any two elements of F, we have,

$$a\alpha + b\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0)$$
$$= (aa_1, aa_2, 0) + (bb_1, bb_2, 0)$$
$$= (aa_1 + bb_1, aa_2 + bb_2, 0) \in W$$

Since, $(aa_1 + bb_1), (aa_2 + bb_2) \in F$.

$\therefore$ W is a vector sub-space of V_3 over F.

Example Let $V = \{(x, y, z) : x, y, z \in R\}$ where R is the field of real numbers. Show that, if $W = \{(x, y, z) : x - 3y + 4z = 0\}$, then it is a sub-space of V over R.

Sol^n:→ Let $\alpha, \beta \in W$; then we may write

$$\alpha = (3y_1 - 4z_1, y_1, z_1) \text{ and } \beta = (3y_2 - 4z_2, y_2, z_2)$$

If $a, b \in R$, then we have,

$$a\alpha + b\beta = a(3y_1 - 4z_1, y_1, z_1) + b(3y_2 - 4z_2, y_2, z_2)$$

$$= (3ay_1 - 4az_1, ay_1, az_1) + (3by_2 - 4bz_2, by_2, bz_2)$$

$$= [3(ay_1 + by_2) - 4(az_1 + bz_2), ay_1 + by_2, az_1 + bz_2]$$

$$= [3l - 4m, l, m] \in W$$

since $l = ay_1 + by_2 \in R$ and $m = az_1 + bz_2 \in R$

$\therefore$ $a, b \in R$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

$\therefore$ W is a vector sub-space of V in R.

Example:→ If a_1, a_2, a_3 be fixed elements of a field F, then the set W of all ordered triads (x_1, x_2, x_3) of element of F such as,

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

is a sub-space of V_3 in F.

Sol^n:→ Let $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$

Then $a_1x_1 + a_2x_2 + a_3x_3 = 0 \longrightarrow (1)$

$a_1y_1 + a_2y_2 + a_3y_3 = 0 \longrightarrow (2)$

For $x_1, x_2, x_3, y_1, y_2, y_3 \in F$

Let a and b be any two elements of F, then

$$a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3)$$

$$= (ax_1, ax_2, ax_3) + (by_1, by_2, by_3)$$

$$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \in W$$

Since, $a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3)$

$$= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3)$$

$$= a(0) + b(0)$$

$$= 0 \quad \text{(by 1 and 2)}$$

Hence W is a sub-space of V_3 in F.

Linear Combinations: Let V be a vector space over the field F. If $\vec{\alpha}_1, \vec{\alpha}_2, \ldots \vec{\alpha}_n \in V$, then any vector $\vec{\alpha}$ said to be a linear combination of the vectors $\vec{\alpha}_1, \vec{\alpha}_2, \ldots \vec{\alpha}_n$ if

$$\vec{\alpha} = a_1\vec{\alpha}_1 + a_2\vec{\alpha}_2 + \cdots\cdots + a_n\vec{\alpha}_n,$$

where the scalars $a_1, a_2, \ldots a_n \in F$.

example: Let $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \in V$, $\alpha \in V$

$$\therefore \bar{\alpha} = a_1\vec{\alpha}_1 + a_2\vec{\alpha}_2 + a_3\vec{\alpha}_3$$

If $\bar{\alpha}_1 = (1,0,0)$, $\bar{\alpha}_2 = (0,1,0)$, $\bar{\alpha}_3 = (0,0,1) \in V$

$\therefore$ any vector $\bar{\alpha} = (3,5,1)$ can be expressed as,

$$(3,5,1) = 3(1,0,0) + 5(0,1,0) + 1(0,0,1)$$
$$= (3,5,1)$$

$\therefore$ $(3,5,1)$ is a linear combination of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$

example: $\bar{\alpha} = (1,1)$, $\bar{\beta} = (1,2)$

$$\therefore (7,14) = a\bar{\alpha} + b\vec{\beta} = a(1,1) + b(1,2)$$
$$= (a+b, a+2b)$$

$$\therefore a+b = 7,$$
$$(-)\ a+2b = 14.$$
$$-b = -7$$
$$\Rightarrow \boxed{b=7}\ ; \ \boxed{a=0}$$

Example: Express $(-1,2,4)$ as a linear combination of $\bar{\alpha} = (-1,2,0)$, $\bar{\beta} = (0,-1,1)$, $\bar{\gamma} = (3,-4,2)$ in the vector space V_3 of real number.

soln: Let a, b, c three scalar in real number such that,

$$(-1,2,4) = a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma}$$
$$= a(-1,2,0) + b(0,-1,1) + c(3,-4,2)$$
$$= (-a+3c, 2a-b-4c, b+2c)$$

$$\therefore \left.\begin{aligned} -a+3c &= -1 \\ 2a-b-4c &= 2 \\ b+2c &= 4 \end{aligned}\right\} \text{on solving, we get } \boxed{a=4}, \boxed{b=2}, \boxed{c=1}$$

$$\therefore (-1,2,4) = 4(-1,2,0) + 2(0,-1,1) + 1(3,-4,2)$$

Linearly dependent Vectors :→ Let V be the vector space over the field F, a finite sub-set $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots \bar{\alpha}_n\}$ of vectors of V said to be linearly dependent, if there exists scalars $a_1, a_2, \cdots a_n \in F$, not of all then zero, such that,

$$a_1\vec{\alpha}_1 + a_2\vec{\alpha}_2 + \cdots\cdots + a_n\bar{\alpha}_n = \bar{0}$$

Linearly Independent Vectors :→ Let V be the vector space over the field F, a finite subset $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots \bar{\alpha}_n\}$ of vectors of V said to be linearly independent, if where scalars, $a_1 = a_2 = \cdots = a_n = 0 \in F$, such that,

$$a_1\bar{\alpha}_1 + a_2\bar{\alpha}_2 + \cdots\cdots + a_n\bar{\alpha}_n = \bar{0}$$

Example :→ Show that the vectors $\{(2,-3,1),(3,-1,5),(1,-4,3)\}$. linearly independent in $V_3(R)$.

Sol^n :→ Let a, b, c be three scalars in real numbers such that,

$$a(2,-3,1) + b(3,-1,5) + c(1,-4,3) = (0,0,0)$$

$$\Rightarrow [2a + 3b + c, -3a - b - 4c, a + 5b + 3c] = (0,0,0)$$

$$\therefore \quad 2a + 3b + c = 0$$
$$-3a - b - 4c = 0$$
$$a + 5b + 3c = 0$$

Note :→

$$AX = 0$$

$$\begin{bmatrix} 2 & 3 & 1 \\ +3 & +1 & +4 \\ 1 & 5 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

If rank of A is = no. of unknown then the system have only sol^n

$$a = b = c = 0$$

$|A| \neq 0$.

Let $AX = 0$

where $A = \begin{bmatrix} 2 & 3 & 1 \\ +3 & +1 & +4 \\ 1 & 5 & 3 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

now $|A| = 2(3 - 20) - 3(9 - 4) + 1(15 - 1)$

$= -35$

$\neq 0$.

$\therefore$ rank of $A = 3$ = no. of unknowns.

Hence, $a = b = c = 0$ is the only sol^n, Thus the given system is linearly independent.

Question :→ Examine wheather the sets of vectors are linearly dependent or linearly independent. $\{(1,0,1), (1,1,0), (1,-1,1), (1,2,-3)$.

Sol^n :→ Let a, b, c, d be the 4 scalars of real numbers,

$$a(1,0,1) + b(1,1,0) + c(1,-1,1) + d(1,2,-3) = (0,0,0)$$

$$\Rightarrow (a + b + c + d, \; b - c + 2d, \; a + c - 3d) = (0,0,0)$$

$$\therefore \quad a + b + c + d = 0 \longrightarrow (1)$$
$$b - c + 2d = 0 \longrightarrow (2)$$
$$a + c - 3d = 0 \longrightarrow (3)$$

Let, $AX = 0$, let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 1 & -3 \end{bmatrix}$, $X = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

Now, $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -1 \neq 0$

$\therefore$ rank of the matrix is 3 which is less than the unknowns,

Hence, the equation will possess a non-zero solution,

Let $d = 1$

$\therefore \quad a + b + c + 1 = 0 \longrightarrow (3)$

$b + c + 2 = 0 \longrightarrow (4)$

$a + c - 3 = 0 \longrightarrow (5)$

on solving the equations (3), (4), (5) we get,

$b = -4$, $c = -2$, $a = 5$

Hence, the given set is linearly dependent.

Linear span :→ Let V be a vector space over the field F and S be any non-empty sub-set of V. Then the linear span of S is defined as the set of all linear combinations of finite sets of elements of S. It is denoted by $L(S)$.

Thus we have,

$L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n : \alpha_i \in S, a_i \in F, i = 1, 2, \ldots n\}$.

Basis and dimension of a Vector Space :→ Let V be a vector space over the field F and S be a sub-set of $V(F)$. Such that

(i) S is a set of linearly independent vectors in V and,

(ii) $L(S) = V$, that is each vector in V is a linear combination of a finite number of elements of S, then S is called a basis set or simply a basis of V.

e.g consider the set $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ in V_3 over the real numbers.

this set is linearly independent. Also B spans V_3, because any vector (a_1, a_2, a_3) of V_3 can be written as a linear combination of the vectors of B. i.e

$$(a_1, a_2, a_3) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$$

It is called a standard basis R^3.

Dimension :→ The number of elements in any basis set of a finite dimensional vector space $V(F)$ is called the dimension of the vector space and is denoted by $\dim V$.

$V_n(F)$ is n-dimensional, if its basis contains n elements.

The dimension of the vector space R^2 is 2, since.

$$B = \{(1,0), (0,1)\} \text{ is a basis.}$$

The vector space R^3 is of dimension 3, as,

$$B = \{(1,0,0), (0,1,0), (0,0,1)\}$$

Note:- A vector space V is said to be finite dimensional or finitely generated, if there exist a finite sub-set S of V such that $L(S) = V$. Otherwise, the vector space is infinite dimensional.

Question:- Show that the vectors (1,2,1), (2,1,0), (1,−1,2) form a basis of the vector space V_3 over the field of real numbers.

Sol:- We know that if V(F) be a finite dimensional vector space of dimension n, then any set of n linearly independent vectors in V forms a basis of V.

Now the set {(1,0,0), (0,1,0), (0,0,1)} forms a basis of the vector space V_3 over the field of real numbers. Hence its dimension is 3. If we can show that the set,

$$S = \{(1,2,1), (2,1,0), (1,-1,2)\}$$

is linearly independent, then S will also form a basis of V_3.

Now, $a_1(1,2,1) + a_2(2,1,0) + a_3(1,-1,2) = (0,0,0)$

$$\Rightarrow (a_1 + 2a_2 + a_3,\ 2a_1 + a_2 - a_3,\ a_1 + 2a_3) = (0,0,0)$$

$$\therefore \quad a_1 + 2a_2 + a_3 = 0$$
$$2a_1 + a_2 - a_3 = 0$$
$$a_1 + 2a_3 = 0.$$

The coefficient matrix is $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} = A$ (say)

Now, $|A| = -9 \neq 0$. Thus the rank of A is 3 = no. of unknown. Hence solving these equation, we have the only solⁿ,

$$a_1 = a_2 = a_3 = 0.$$

Therefore the set S is linearly independent. Hence it forms a basis of V_3 in the field of the real numbers.

Functions:-

We can pass from one vector space to another by means of some functions possessing certain linearity property and are known as linear transformations.

A function consists of the following

a) a set V, which is called domain of the function.

b) a set W, which is called co-domain of the function.

c) a rule f, which associtetes each element v of V, a single element $f(v)$ of W.

Linear Transformation :→ Let V and W be vector spaces over the field F. A linear transformation from V into W is a function f from V into W such that,

$$f(c\alpha+\beta) = cf(\alpha)+f(\beta) \longrightarrow ① \quad \Big| \; f(a\alpha+b\beta) = af(\alpha)+bf(\beta).$$

for all α, β in V and all scalars c in F.

This is also called linearity property.

Question :→ Show that the transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(x,y) = (x-y,\ x+y,\ y)$$

is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Solⁿ :→ Let $\alpha = (x_1, y_1)$ and $\beta = (x_2, y_2) \in \mathbb{R}^2$

$\therefore\ T(\alpha) = T(x_1, y_1) = (x_1-y_1,\ x_1+y_1,\ y_1)$

and $T(\beta) = T(x_2, y_2) = (x_2-y_2,\ x_2+y_2,\ y_2)$

Also, $a, b \in \mathbb{R}$. Then $a\alpha + b\beta \in \mathbb{R}^2$

and
$$\begin{aligned}
T(a\alpha+b\beta) &= T[a(x_1, y_1) + b(x_2, y_2)] \\
&= T[ax_1+bx_2,\ ay_1+by_2] \\
&= [ax_1+bx_2-ay_1-by_2,\ ax_1+bx_2+ay_1+by_2,\ ay_1+by_2] \\
&= [a(x_1-y_1)+b(x_2-y_2),\ a(x_1+y_1)+b(x_2+y_2),\ ay_1+by_2] \\
&= a(x_1-y_1,\ x_1+y_1,\ y_1) + b(x_2-y_2,\ x_2+y_2,\ y_2) \\
&= aT(\alpha) + bT(\beta).
\end{aligned}$$

Therefore, T is a linear transformation.

Zero Transformation :→ Let V and W be two vector spaces over the same field F.

The function $f: V \to W$ defined by,

$f(\alpha) = \vec{0}$ (zero vector of W); for all $\alpha \in V$.

is a linear transformation from V into W.

Let $\alpha, \beta \in V$. and $a, b \in F$

Then, $a\alpha + b\beta \in V$

Now we have, $f(a\alpha+b\beta) = \bar{0} = a\cdot\bar{0} + b\cdot\bar{0} = af(\alpha) + b\cdot f(\beta)$.

$\therefore f$ is a linear transform from V into W.

This transformation is called a zero transformation.

Properties of linear Transformation:→ Let T be a transform from a vector space V into a vector space W over the field F. Then

1. $T(\bar{0}) = \bar{0}'$, where $\bar{0}$ and $0'$ are the zero vectors of V and W respectively.
2. $T(-\alpha) = -T(\alpha)$, for all $\alpha \in V$.
3. $T(\alpha - \beta) = T(\alpha) - T(\beta)$, for all $\alpha, \beta \in V$.

Singular and non-singular:→ Let $T: V \to W$ be a linear transformation for two vector spaces V and W under the same field F. The set of images of the elements of V under the transformation T is said to be the image of T and is denoted by $I_m(T)$

If a linear transformation $T: V \to W$ be such that the image of some non-zero vector $\in V$ under T is $\bar{0}' \in W$, then the linear transformation is called singular.

Thus T is a non-singular transformation if only $\bar{0} \in V$ maps into $0' \in W$ under T, that is if null space of T consists of only zero vector.

Example:→ If $T: V_3 \to V_1$ and $T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$, then show that T is not a linear transformation.

Sol^n:→ Let $x = y = (1, 0, 0)$

$$\begin{aligned} \text{then, } T(x+y) &= (x_1+y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2 \\ &= (1+1)^2 + (0+0)^2 + (0+0)^2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{while, } T(x) + T(y) &= x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 \\ &= 1^2 + 0^2 + 0^2 + 1^2 + 0^2 + 0^2 \\ &= 2. \end{aligned}$$

$\because T(x+y) \neq T(x) + T(y)$, then T is not a linear transformation.

Representation of Linear Transformation by matrices:→

Let V be an n-dimensional vector space over the field F and W b an m dimensional vector space over the same field.

Let, $B_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V and
$B_2 = \{\beta_1, \beta_2, \ldots, \beta_m\}$ " " " " for W.

Let T be any transformation from V into W, then each of the n vectors $T(\alpha_j)$, $j = 1, 2, \ldots n$, can be expressed uniquely as a linear combination of the elements of B_2. Let

$$T(\alpha_j) = a_{1j}\beta_1 + a_{2j}\beta_2 + \cdots + a_{mj}\beta_m \; ; \; j = 1, 2, \ldots n \quad \longrightarrow (1)$$

The Scalars $a_{1j}, a_{2j}, \ldots\ldots, a_{mj}$ are the co-ordinates of $T(\alpha_j)$ and the transformation T is determined by the mn Scalars a_{ij} according to 1.

The matrix $A = [a_{ij}]_{m\times n}$ is called the matrix of T relative to the pair of ordered bases B_1 and B_2.

Example: Let T be a linear transformation of R^2 into itself that maps $(1,1)$ to $(-2,3)$ and $(1,-1)$ to $(4,5)$. Determine the matrix representing T with respect to the base $\{(1,0),(0,1)\}$.

Sol^n:→ we are to determine the effect of T when applied to $(1,0)$ and $(0,1)$.

Now, $(1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$

and $(0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$

∵ T is the linear transformation, we have

$$\begin{aligned}
T(1,0) &= T\left[\tfrac{1}{2}(1,1) + \tfrac{1}{2}(1,-1)\right] \\
&= \tfrac{1}{2}T(1,1) + \tfrac{1}{2}T(1,-1) \\
&= \tfrac{1}{2}(-2,3) + \tfrac{1}{2}(4,5). \\
&= \tfrac{1}{2}(2,8) \\
&= (1,4) \\
&= 1\,(1,0) + 4\,(0,1) \quad \longrightarrow (1)
\end{aligned}$$

Also,

$$\begin{aligned}
T(0,1) &= T\left[\tfrac{1}{2}(1,1) - \tfrac{1}{2}(1,-1)\right] \\
&= \tfrac{1}{2}T(1,1) - \tfrac{1}{2}T(1,-1) \\
&= \tfrac{1}{2}(-2,3) - \tfrac{1}{2}(4,5) \\
&= \tfrac{1}{2}(-6,-2) \\
&= (-3,-1) \\
&= -3\,(1,0) - 1\,(0,1) \quad \longrightarrow (2)
\end{aligned}$$

∴ from (1) and (2) we have the representative matrix as,

$$\begin{bmatrix} 1 & -3 \\ 4 & -1 \end{bmatrix} \text{Ans}$$

Example :→ Let T be the linear operator on R^3 defined by,

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

Find the matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where,

$\alpha_1 = (1,0,1)$, $\alpha_2 = (-1,2,1)$, $\alpha_3 = (2,1,1)$

Sol^n :→ From the given definition of T, we have,

$$T(\alpha_1) = T(1,0,1) = (4,-2,3)$$

Now, we are to express $(4,-2,3)$ as a linear combination of the vectors of the basis $\{\alpha_1, \alpha_2, \alpha_3\}$.

$$\therefore (a,b,c) = x\alpha_1 + y\alpha_2 + z\alpha_3$$

$$= x(1,0,1) + y(-1,2,1) + z(2,1,1)$$

$$= (x-y+z,\ 2y+z,\ x+y+z)$$

$$\therefore \left.\begin{aligned} x-y+z &= a \\ 2y+z &= b \\ x+y+z &= c \end{aligned}\right\} \to (1)$$

→ Putting, $a=4$, $b=-2$ and $c=3$ ~~we get~~ on solving we have,

$$x = \frac{17}{4},\ y = -\frac{3}{4},\ z = -\frac{1}{2}$$

$$\therefore T(\alpha_1) = T(1,0,1) = \frac{17}{4}\alpha_1 - \frac{3}{4}\alpha_2 - \frac{1}{2}\alpha_3.$$

Similarly $T(\alpha_2) = T(-1,2,1) = (-2,4,9)$, from the definition.

Putting $a=-2$, $b=-2$, $C=3$ in (1), we have,

$$x = \frac{35}{4},\ y = \frac{15}{4},\ z = -\frac{7}{2}$$

$$\therefore T(\alpha_2) = T(-1,2,1) = \frac{35}{4}\alpha_1 + \frac{15}{4}\alpha_2 - \frac{7}{2}\alpha_3.$$

Finally $T(\alpha_3) = T(2,1,1) = (7,-3,4)$.

Putting $a=7$, $b=-3$ and $c=4$ in (1) we get

$$x = \frac{11}{2},\ y = -\frac{3}{2},\ z = 0.$$

$$\therefore T(\alpha_3) = T(2,1,1) = \frac{11}{2}\alpha_1 - \frac{3}{2}\alpha_2 + 0\cdot\alpha_3.$$

Thus the matrix of the transformation, T is,

$$\begin{bmatrix} \frac{17}{4}, & \frac{35}{4}, & \frac{11}{2} \\ -\frac{3}{4} & \frac{15}{4} & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{7}{2} & 0 \end{bmatrix}$$

Example :→ If the matrix of a linear transformation T on $V_2(C)$ with respect to the ordered basis,

$B = \{(1,0),(0,1)\}$ be $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then find the matrix of T with respect to the ordered basis. $B' = \{(1,1),(-1,1)\}$.

Sol^n:→ we are given that,

$$[T]_B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad T(1,0) = 1(1,0) + 1(0,1) = (1,1)$$

$$T(0,1) = 1(1,0) + 1(0,1) = (1,1)$$

If $a, b \in V_2$, then we have,

$$(a,b) = a(1,0) + b(0,1)$$

$$\therefore T(a,b) = T[a(1,0) + b(0,1)]$$

$$= aT(1,0) + bT(0,1)$$

$$= a(1,1) + b(1,1)$$

$$= (a+b, a+b)$$

This represents the linear transformation T.

Now let us find the matrix of T with respect to $B' = \{(1,1),(-1,1)\}$.

Now, $T(1,1) = (2,2)$

$$\therefore (2,2) = x(1,1) + y(-1,1) = (x-y, x+y)$$

$$\therefore \left.\begin{aligned} x-y &= 2 \\ x+y &= 2 \end{aligned}\right\} \text{ which gives } x = 2, y = 0$$

$$\therefore T(1,1) = 2(1,1) + 0(-1,1).$$

Also, $T(-1,1) = (0,0) = 0(1,1) + 0(-1,1)$.

$$\therefore [T]_{B'} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Ans